Damping Identification with the Morlet-Wave

Janko Slavič, Miha Boltežar

Laboratory for Dynamics of Machines and Structures, Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, 1000 Ljubljana, Slovenia

Cite as:

J. Slavič and M. Boltežar Damping identification with the Morlet-wave. Mechanical Systems and Signal Processing, 2011, Volume 25, Issue 5, July 2011, Pages 1632-1645. DOI: 10.1016/j.ymssp.2011.01.008

Abstract

In the past decade damping-identification methods based on the continuous wavelet transform (CWT) have been shown to be some of the best methods for analyzing the damping of multi-degree-of-freedom systems. The CWT methods have proven themselves to be resistant to noise and able to identify damping at closely spaced natural frequencies. However, with the CWT-based techniques, the CWT needs to be obtained on a two-dimensional, time-frequency grid, and they are, therefore, computationally demanding. Furthermore, the CWT is susceptible to the edge effect, which causes a non-valid identification at the start and the end of the time-series.

This study introduces a new method, called the Morlet-wave method, where a finite integral similar to the CWT is used for the identification of the viscous damping. Instead of obtaining the CWT on a two-dimensional grid, the finite integral needs to be calculated at one time-frequency point, only. Then using two different integration parameters, the damping ratio can be identified. A complete mathematical background of the new, Morlet-wave, damping-identification method is given and this results in a root-finding or a closed-form solution.

The presented numerical experiments show that the new method has a similar performance to the CWT-based damping-identification methods, while the method is numerically, significantly less demanding, completely avoids the edge effect, and the procedure is straight forward to use.

Email address: janko.slavic@fs.uni-lj.si (Janko Slavič) *URL:* www.fs.uni-lj.si/ladisk (Janko Slavič)

Preprint submitted to Mechanical Systems and Signal Processing

Keywords: Morlet, wave, wavelet, continuous wavelet transform, damping identification, noise, close modes, exact,

1. Introduction

Oscillating mechanical systems involve the exchange of kinetic and potential energy, while the dissipation of energy, caused by damping, forces the oscillations to die out. Compared to the stiffness and mass properties, however, these damping parameters are more difficult to identify. The factors affecting damping include dry friction, viscous friction in fluids, and friction on the atomic level. Due to the broad range of damping influences a number of simplified models were developed. One of the most widely used models, the viscous damping model, assumes the damping force is proportional to the velocity. Another widely used model is the structural damping model, where the dissipation of energy in a single oscillation is independent of the frequency [1]. A generalization of the different damping models is possible with the equivalent viscous damping model. In this research the damping will be discussed in terms of the damping ratio, i.e., the fraction of critical damping.

Pradina *et al.* [2] studied the performance of different approaches to determining linear viscous damping: from a closed-form solution, identification methods based on inverting the matrix of receptances, energy expressions developed from single-frequency excitation and responses as well as first-order perturbation methods. The experimental identification of modal damping ratios can be based on forced vibration, free vibration or ambient vibration tests [3]; this research focuses on the continuous wavelet transform (CWT) based techniques of damping identification from the free vibration response as a consequence of a impact excitation.

In the past decade the CWT-based damping-identification methods have proved to be some of the more promising damping-identification methods and were extensively researched by Staszewski [1, 4], Ruzzene *et al.* [5], Lamarque *et al.* [6, 7], and Ghanem and Romeo [8]. This was followed by research focused on noise and enhancements to the edge effect¹ by Slavič and Boltežar [9, 10]. Later studies, by Lardies *et al.* [11, 12] and Argoul *et al.* [13, 14, 15], among others, looked at modal identification with the CWT [16]. Recently, Chen *et.al* [17] researched guidelines for system identification with the Morlet mother wavelet.

¹Also called end-effect.

Some recent research has compared the CWT methods to the Hilbert-Huang transform methods [18] and also to Gabor analysis [19]. The use of the CWT for damping-identification uses pattern-search updating for the correction of the identification results [20] and the frequency-slice wavelet transform for the transient vibration response analysis [21, 22].

The CWT methods have proven to be resistant to noise and can identify damping at closely spaced natural frequencies [1, 9, 10]. Usually, the CWT methods require the following steps: the calculation of the CWT transform at a time-frequency grid, followed by the ridge and skeleton detection phase, and, finally, after the edge effected region has been removed from the identification area, the damping ratio can be identified from the logarithmic decay. The CWT transform at a dense time-frequency grid can, however, be numerically very demanding. Furthermore, the edge effect requires a manual selection of the time-window appropriate for the damping-identification, and as was shown by Boltežar and Slavič [10], up to 80% of the signal can be lost to the edge effect.

This research proposes a new method, called the Morlet-Wave (MW) dampingidentification method, which significantly decreases the numerical load to one time-frequency point that needs to be evaluated at two parameter sets and completely avoids the edge effect. The basic idea is presented in Section 2, while the basics of the continuous wavelet transform are given in Section 3 and the MW damping-identification method is presented in Sections 4. This is followed by the numerical experiment in Section 5, where the numerical investigation of the presented method is given. The summary of the Morlet-wave method is given in Section 6. The conclusions are given in Section 7, and the Appendix presents the details of the MW mathematical deduction.

2. Basic Idea

Figure 1 shows the procedure for the damping-identification based on the CWT: the signal first needs to be transformed at a relatively dense timefrequency grid, which represents the numerically most demanding step of the damping identification. The second step of the CWT-based dampingidentification is the ridge detection. The ridges represent the frequency content of the analyzing signal with a high energy density, which is dependent on the time. Staszewski [1] discussed three methods of ridge detection: the *cross-sections method*, with selected (constant) frequency, the *amplitude method*, which is based on the maxima of the CWT; and the *phase method*, which is based on matching the angular velocity of the CWT with the angular velocity of the wavelet function. The values of the CWT that are restricted to the ridge are called the *skeleton*. The third step in the CWTbased damping-identification is the edge effect consideration. Boltežar and Slavič [10] showed that the edge effect depends a lot on the identified damping ratio. The last step of the CWT-based damping-identification is to identify the damping ratio using the logarithmic decay of the skeleton.

The Wave-method procedure shown in Figure 1 starts by selecting the natural frequency with which the damping ratio should be identified. The natural frequency is considered to be constant with time². In the second step the CWT similar finite integral is calculated with two different wave parameters (shown as wave 1 and wave 2 in Figure 1). The resulting finite integral hides the unknown parameters of the oscillating sinusoidal: the amplitude, the phase and the damping ratio. In Section 4.2 this study shows that the unknown damping-ratio can be identified using a simplified closed-form solution, or the exact solution that requires root-finding.

3. Continuous Wavelet transform

The continuous wavelet transform (CWT) of the measured signal $f_m(t) \in L^2(\mathbf{R})$ is defined as:

$$Wf_m(u,s) = \int_{-\infty}^{+\infty} f_m(t) \,\psi_{u,s}^*(t) \,\mathrm{d}t,\tag{1}$$

where u and s are the translation and scalation parameters, respectively [23], and $\psi_{u,s}^*(t)$ is the translated-and-scaled complex conjugate of the basic/mother wavelet function $\psi(t) \in L^2(\mathbf{R})$. The wavelet function is a normalized function (i.e., the norm is equal to 1) with an average value of zero [24].

The normalized Morlet mother wavelet function [24, 25] is defined as:

$$\psi(t,\eta) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{t^2}{2}} e^{i\eta t}.$$
(2)

where η is the modulation frequency and $\frac{1}{\sqrt[4]{\pi}}$ is used for the normaliza-

²Using CWT-based damping-identification techniques on real experiments showed that the frequency of oscillation rarely changes significantly with time [9]. Furthermore, the goal is to identify the damping from as short a signal as possible (i.e., 30 oscillations) giving another reason to assume a constant frequency



Figure 1: Comparison of the CWT-based damping-identification procedure with the Morlet-wave-method procedure.

tion [25]. The scaled-and-translated wavelet function is:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right). \tag{3}$$

For this study the important properties of the Morlet wavelet functions are [25]:

$$\overline{\omega}_{u,s} = \frac{\eta}{s}$$
 Center frequency. (4)

$$\sigma_{t_{u,s}} = \frac{s}{\sqrt{2}}$$
 Time spread. (5)

$$\sigma_{\omega_{u,s}} = \frac{1}{\sqrt{2}s} \qquad \text{frequency-spread.} \tag{6}$$

A very useful property of the CWT is its linearity:

$$\left(W\sum_{i=1}^{N}\alpha_{i} x_{i}\right)(u,s) = \alpha_{i}\sum_{i=1}^{N} (W x_{i})(u,s),$$

$$(7)$$

which makes it possible to analyze each i^{th} component x_i of a multi-component signal $\sum_{i=1}^{N} \alpha_i x_i$, where α_i is a constant. The closely spaced frequencies ω_1 and ω_2 can be successfully identified if the frequency-spread is smaller than their difference [24]:

$$\max(\sigma_{\omega_{u,s_1}}, \sigma_{\omega_{u,s_2}}) < |\omega_1 - \omega_2|. \tag{8}$$

Recently, Chen *et.al* [17] researched the Morlet wavelet parameter selection on the closely spaced frequencies.

4. Morlet-Wave Damping-Identification Method

4.1. Equivalent Viscous Damping Model

Damping mechanisms include friction on the atomic/molecular level, dry friction, viscous friction in fluids, etc., and so it is often difficult to describe in detail the real physical background using mathematical means. As a consequence of this, a number of simplified models were developed (dry friction, viscous, hysteretic, and others). Of these models, the model of viscous damping is the most widely used, it assumes that the damping force is proportional to the velocity of oscillation; and so it follows that the work done by one oscillation cycle depends on the frequency of oscillation. In structural damping the work done in one cycle is independent of the oscillation frequency and the dissipation of vibrational energy is proportional to the square of the amplitude [26].

To overcome the shortcomings of the different models the model of equivalent viscous damping can be used [27]. Therefore, viscous damping in terms of the damping ratio (i.e. the fraction of critical damping) is used in this research. As a result, the damping matrix can be assumed to be proportional to the mass or stiffness matrix, so the system of differential equations can be uncoupled [9, 26].

4.2. Theoretical Background of the Morlet-Wave Damping-Identification Method The measured (viscously) damped signal $f_m(t)$ is defined as

$$f_m(t) = X e^{-\delta \omega t} \cos(\omega_d t - \varphi); \qquad 0 \le t \le T, \tag{9}$$

where X is the amplitude, δ is the damping ratio, φ is the phase, T is the time-length of the analyzed mode, and ω and ω_d are the undamped and damped oscillating frequencies, respectively. For this study the usual assumption for lightly damped dynamical systems was used, *i.e.*, $\omega_d = \omega \sqrt{1 - \delta^2} \approx \omega$.

The MW damping-identification method is based on a finite integral that is similar to the CWT using the Morlet wavelet:

$$I = \int_0^T f_m(t) \,\psi_{u,s}^*(t) \,\mathrm{d}t,$$
 (10)

where the translation parameter u is defined as u = T/2. The scale s influences the time-spread of the CWT via Eq. (5). As was discussed by Boltežar and Slavič [10], the time-spread of the CWT influences the resistance of the damping-identification to noise. For the subsequent mathematical manipulation it is reasonable to relate the time-length of the analyzed signal T to the time-spread: $T = n \sigma_{t_{u,s}} = n s / \sqrt{2}$, where n is called the time-spread parameter. It follows that the scale is defined as:

$$s = \frac{\sqrt{2}T}{n}.\tag{11}$$

In general, the duration T is arbitrary; however, when the theoretical integration limits of Eq. (1) go from ∞ to limited values, the integration errors are more pronounced; this is particularly so for the phase error [10]. To keep this error small and for the sake of later mathematical manipulations, T is limited to integer multiples of the oscillation period of the damped signal $\Delta T = 2\pi/\omega$:

$$T = k \frac{2\pi}{\omega},\tag{12}$$

where k defines the number of oscillations of frequency ω in the measured signal $f_m(t)$ (9). The damping identification starts with selecting the natural frequency of interest ω , followed by selecting the proper value for integer k. Clearly, the total length of the measured signal $f_m(t)$ needs to be longer then the analyzed time T.

From Eqs. (11) and (12) it follows that:

$$s = \frac{2\sqrt{2}\pi k}{\omega n}.$$
(13)

The center frequency (4) of the CWT needs to match the oscillating frequency of the measured signal $\overline{\omega}_{u,s} = \omega$, and using Eqs. (12) and (13) the frequency of modulation η is defined as:

$$\eta = \omega \, s = \frac{2\sqrt{2}\,\pi\,k}{n}.\tag{14}$$

Rewriting Eq.(10):

$$I = \int_0^T f_m(t) \left(\frac{1}{\sqrt[4]{\pi}\sqrt{s}} e^{-\frac{(t-T/2)^2}{2}} e^{-i\eta \frac{t-T/2}{s}} \right) dt$$
(15)

and then by using the parameters n, k and ω , only:

$$I(n,k,\omega) = \int_{0}^{k\frac{2\pi}{\omega}} f_{m}(t) \left(\frac{1}{\sqrt[4]{\pi}\sqrt{\frac{2\sqrt{2}\pi k}{\omega n}}} e^{-\frac{(\frac{t-k\frac{2\pi}{\omega}/2}{2\sqrt{2}\pi k})^{2}}{2}} e^{-i\frac{2\sqrt{2}\pi k}{n}} \frac{t-k\frac{2\pi}{\omega}/2}{\frac{2\sqrt{2}\pi k}{\omega n}} \right) dt,$$
(16)

Using the assumptions $\delta \leq \frac{n^2}{8\pi k}$ (A.13) the amplitude of $I(n, k, \omega)$ Eq.(16) is approximated with Eq.(A.19) (for details, see Appendix A):

$$|I(n,k,\omega)| \approx X \sqrt[4]{2\pi^3} \sqrt{\frac{k}{n\omega}} e^{\frac{\pi k \delta \left(4\pi k \delta - n^2\right)}{n^2}} \left(\operatorname{Erf}\left(\frac{2\pi k \delta}{n} + \frac{n}{4}\right) - \operatorname{Erf}\left(\frac{2\pi k \delta}{n} - \frac{n}{4}\right) \right),$$
(17)

where X is the unknown amplitude and δ is the unknown damping ratio (the phase was reduced by obtaining the amplitude of $I(n, k, \omega)$).

4.3. Exact Morlet-Wave Damping-Identification Method

The unknown amplitude of oscillation X is reduced by dividing two $|I(n, k, \omega)|$ functions at different time-spread parameters n_1 and n_2 :

$$M(n_1, n_2, k, \omega) = \frac{|I(n_1, k, \omega)|}{|I(n_2, k, \omega)|},$$
(18)

resulting in:

$$M(n_1, n_2, k, \omega) = e^{\frac{4\pi^2 k^2 \delta^2 \left(n_2^2 - n_1^2\right)}{n_1^2 n_2^2}} \sqrt{\frac{n_2}{n_1}} \underbrace{\frac{\operatorname{Erf}\left(\frac{2\pi k\,\delta}{n_1} + \frac{n_1}{4}\right) - \operatorname{Erf}\left(\frac{2\pi k\,\delta}{n_1} - \frac{n_1}{4}\right)}_{G}}_{G}$$
(19)

All the parameters except the damping ratio δ are known, and obtaining the numerical values of $M(n_1, n_2, k, \omega)_{Num}$ by numerical integration Eq.(16) the Eq. (19) can be solved in a non-algebraic way by finding the numerical solution for δ .

4.4. Simplified Closed-Form Morlet-Wave Damping-Identification Method

Further simplifications are possible if $n_1 > 10$ and $n_1 < n_2$ when the part G in Eq. (19) is approximated by $G \approx 1$ and an algebraic solution for the unknown δ is possible:

$$\delta_{Morlet} = \frac{n_1 n_2}{2 \pi \sqrt{k^2 n_2^2 - k^2 n_1^2}} \sqrt{\ln\left(\sqrt{\frac{n_1}{n_2}} M(n_1, n_2, k, \omega)_{Num}\right)}.$$
 (20)

As will be discussed later, the simplified, closed-form method has a much smaller application range than the exact method.

4.5. Parameter Selection

In this section the selection of the parameters n_1 , n_2 , and k of the exact MW damping-identification method will be discussed. For the numerical solution of Eq. (19) it is important that the sensitivity of $M(n_1, n_2, k, \omega)$ to the damping ratio δ is high; mathematically the sensitivity is obtained by differentiating $M(n_1, n_2, k, \omega)$ with respect to δ . Figure 2 shows the sensitivity of $M(n_1, n_2, k, \omega)$ to δ at a typical parameter δ, k, n_1 , or n_2 . From a numerical investigation the sensitivity is easily affected by the number of oscillations k taken into account. Furthermore, a low damping ratio decreases the sensitivity severely and becomes the main obstacle to damping-identification at ultra-low damping ratios in the range 10^{-4} to 10^{-6} .

Selection of the time-spread parameters n_1 and n_2 . Regarding the identification sensitivity, the ideal parameter n_1 would be $n_1 = 2.5$; however, a small parameter n_1 limits the maximum damping ratio that can be identified (see Eq. (A.18)), and as a good balance between the sensitivity and the damping-identification range, in this research $n_1 = 5$ was used.

With regard to the identification sensitivity, the ideal parameter n_2 would be very high; however, n_2 is important in identifying the damping at closely spaced modes by Eq. (24). In this research $n_2 = 10$ was used.



Figure 2: Sensitivity of $M(n_1, n_2, k, \omega)$ to the damping ratio δ versus n_1 at typical parameters δ , k. Note: $n_2 = 10$.

Number of oscillations k. The identification sensitivity significantly increases with a larger parameter k; however, a large parameter k decreases the damping-identification range given by Eq. (A.18). From Eq. (A.18) the maximum number of oscillations can be defined as:

$$k_{limit}^{max} \le \frac{n_1^2}{8\,\pi\,\delta}.\tag{21}$$

However, a large k number is not advisable because the frequency-spread defined by Eq. (23) can become too narrow for a real damped signal with a slightly moving frequency of oscillation. For this reason k was limited to k < 300 in this research.

Furthermore, the minimum of the parameter k can be defined from Eq. (9) when the amplitude falls to a defined level. In this research the minimum k was limited with amplitude falling to 30% of the initial level:

$$k_{limit}^{min} \ge -\frac{\log 0.3}{2 \pi \,\delta_e},\tag{22}$$

where δ_e is the estimation of the damping ratio. More details about the selection of k will be given in the numerical section.

Closely spaced frequencies. Closely spaced frequencies can be identified if Eq. (8) is true. Using Eqs.(6,11,12) the frequency-spread of the analyzing

signal at the frequency ω_1 is:

$$\sigma_{\omega_{T/2,s_1}} = \frac{n_2 \,\omega_1}{4 \,\pi \,k} \qquad \text{frequency-spread}(n_1 < n_2). \tag{23}$$

Closely spaced frequencies therefore need to comply with:

$$\max\left(\frac{n_2\,\omega_1}{4\,\pi\,k},\frac{n_2\,\omega_2}{4\,\pi\,k}\right) < |\omega_1 - \omega_2|. \tag{24}$$

5. Numerical Experiment

5.1. Resistance to Noise

In this section the resistance to noise is compared to damping-identification methods using the CWT [1, 9]. The instantaneous Signal-to-Noise Ratio (iSNR) is defined as [9]:

$$iSNR = 10 \log_{10} \left(\frac{var(signal(t))}{var(noise)} \right)$$
(25)

is used to quantify the rate of the noise. The iSNR changes with time because the variance of the damped oscillatory motion decreases with time (the variance of the noise is constant); consequentially, with each oscillation the noise is more pronounced. It is therefore very important that the dampingidentification is made on as small a number of oscillations as possible. In this research the iSNR will always refer to the last oscillation of the sample being analyzed (*i.e.*, the k^{th} oscillation at time T).

In this subsection only the signals of single-degree-of-freedom (SDOF) systems are used. The reason for this is that the iSNR can only be calculated exactly for such a signal. The oscillating frequency of the signal f_m is $\omega_d = 2\pi \operatorname{rad/s} = 1 \operatorname{Hz}$, the amplitude X = 1, and the phase ϕ is randomized for each simulation run. The variance of the Gaussian noise added to the signal is defined via the iSNR, Eq. (25).

The damping-identification parameters are: k = 30 (the length of the signal is 30 oscillations), $n_1 = 5$ and $n_2 = 10$ (the time-spread parameters)³.

The resistance of the Morlet-wave damping-identification to noise was numerically simulated on 2000 samples with the iSNR in the range from 0

³According to the Eq. (A.18) the theoretical maximum damping ratio that can be identified is 0.033. The closer the theoretical limit and the real damping ratio are, the higher the bias of the damping identification. As can be seen from the Figure 3 the bias is approximately 1%; however, if $n_1 = 7$ would be used the bias would fall to approximately 0.25%.

to 40 dB. The error of the damping identification⁴ is shown in Figure 3. The Figure 3b) shows box plots [28] where the box spans the distance from the 0.25 quantile to the 0.75 quantile surrounding the median with lines that extend to span the full dataset. From figure it is clear that the accuracy of the MW method is within 2.5% if the noise rate is smaller than 15 dB and the accuracy is within 10% if the noise is smaller than 5 dB. The sampling frequency for the results in Figure 3 was $10 \omega_d = 10$ Hz.



Figure 3: Error of the damping-identification versus iSNR. a) 2000 random samples, b) box-plot with 400 samples per box.

5.2. Influence of Sampling Frequency

Using the same parameters n_1, n_2, k as in the Section 5.1, but increasing the noise to 5 dB the Figure 4 shows that the resistance to noise can be increased by increasing the sampling frequency: if the sampling frequency is higher than $50 \times \omega_d$ the accuracy at 5 dB of noise is close to 5%. The research proved that with a very high sampling frequency the accuracy of the damping-identification is within 10%, even for signals with very high noise, (e.g. see Figure 5 where a signal with -5dB of noise is shown).

5.3. Range of Damping-Identification

In this section the damping ratio range at which the MW method gives good results will be discussed. As was discussed in Section 4.5, k is usually

 $^{^{4}}$ Using the exact method of Eq. (19)



Figure 4: Error of the damping-identification versus the frequency of the sampling at iSNR=5dB of noise (400 samples per box).



Figure 5: Gray: time-series data with noise (iSNR=-5dB). Black: real part of the Morlet-wave at n = 5.

in the range from 20 to 300. In general, the goal is to identify the damping from as few oscillations as possible, and for a multi-component signal an identification in the range from 20 to 50 oscillations can be considered as

very good. The research on the identification of damping on short signals by Boltežar and Slavič [10] showed a successful damping-identification with the CWT and edge effect reduction methods on signals longer than 75 oscillations (at a damping ratio in the range of 10^{-3}).

Figure 6 shows the damping-identification error for a damping ratio ranging from 10^{-6} to 10^{-2} : the box plots show 400 samples per box where the noise was negligible (120 dB), and the dashed box plots 400 samples per box where the noise was 20 dB^5 .

Figure 6 shows that noisy signals can be analyzed up to a damping ratio of approximately $10^{-3.5}$ and at lower noise levels up to 10^{-5} .



Figure 6: Expected damping-identification error versus the damping ratio at two noise levels (400 samples per box).

5.4. Damping-Identification at Closely Spaced Frequencies

To analyze closely spaced frequencies the signal f_m had two harmonics, the first one was fixed at $\omega_d = 2\pi \operatorname{rad/s=1}$ Hz and the second (the close mode) one ω_{d2} was varied from 0.1 Hz to 3 Hz. The amplitude of both harmonics was X = 1, the phase ϕ was randomized for each of the harmonics and for each simulation run. The damping ratio of the first (the analyzed

⁵In both cases the parameter k was set with regard to Eq. (22) and the frequency of the sampling was $10 \times \omega_d$

harmonic) was $\delta = 0.01$ and the damping of the close mode was $\delta_2 = 0.005$. The noise was negligible (120 dB).

The damping-identification parameters were: k = 30 (the length of the signal is 30 oscillations), $n_1 = 5$ and $n_2 = 10$ (the time-spread parameters). According to Eq. (23) the frequency-spread at ω_d using n_2 is 0.167 Hz. From the numerical research with close modes the damping-identification error at one frequency-spread is within 15%, while at a distance of three frequency-spreads (approximately 0.5 Hz) the error falls to 5%, see Figure 7.



Figure 7: damping-identification of close modes: ω_d is constant, while the close mode ω_{d2} is varied (400 samples per box).

5.5. The Exact Method versus the Closed-Form Method

In this section the exact damping-identification method (Section 4.3) will be compared to the simplified, closed-form, damping-identification method (Section 4.4). Compared to the exact method, the closed-form solution has the advantage that it does not require root-finding; however, the disadvantage is that it requires higher values of n_1 for accurate results. The higher the value n_1 the lower the resistance to noise.

The preferred parameters for the closed-form method $n_1 = 10$, $n_2 = 20$ will be used in this section. The parameter k is defined in similar way to the exact method in the range from k = 30 to k = 300, see Eq. $(22)^6$.

Figure 8 shows that the closed-form MW damping-identification method is, at 20 dB of noise, accurate to within 10%.

So as not to lose the focus of this study, detailed numerical research is not presented here; however, the closed-form solution performs worse on noisy signals than the exact method. A detailed numerical comparison of the exact method and the closed-form method shows that the accuracy reached by the exact method (for a similar number of oscillations k) is reached by the closed-form method when the iSNR is approximately 10 dB higher.



Figure 8: Error of the damping-identification versus iSNR - closed-form MW method (400 samples per box).

6. Summary of the Morlet-Wave Damping-Identification Method

This manuscript introduces a new, Morlet-Wave, damping-identification method. However, because the exact mathematical deductions can distract from the message and the focus, this section only summarizes the Morlet-Wave, damping-identification method.

Imagine a dynamical system with several natural frequencies. The damping identification starts with the acquisition of an impact-response signal $f_m(t)$.

⁶With the selected parameter the G in Eq.(19) at $\delta = 0.01$ equals to 0.9994 and it is reasonable to use the simplified Eq.(20).

As discussed in the Numerical Experiment section, a high sampling frequency increases the damping-identification accuracy and should be at least 10 times higher than the highest natural frequency of interest and the duration of the measurement should be at least 300 oscillations of the lowest natural frequency of interest.

Once the impact response is measured, the damping ratio for a selected natural frequency ω is identified as follows:

1. From the measured time-signal $f_m(t)$, select the part of length T (12):

$$T = k \frac{2\pi}{\omega},\tag{26}$$

where k is an integer value and corresponds to (27):

$$k_{limit}^{min} \ge -\frac{\log 0.3}{2 \pi \,\delta_e}.\tag{27}$$

 δ_e is the expected damping ratio. k should be smaller than 300.

- 2. Select n_1 and n_2 . Typical values for the exact method are $n_1 = 5$, $n_2 = 10$ and for the closed-form method, $n_1 = 10$, $n_2 = 20$.
- 3. Identify $M(n_1, n_2, k, \omega)_{Num}$ (18):

$$M(n_1, n_2, k, \omega)_{Num} = \frac{|I(n_1, k, \omega)|}{|I(n_2, k, \omega)|},$$
(28)

where

$$I(n,k,\omega) = \int_{0}^{k\frac{2\pi}{\omega}} f_{m}(t) \left(\frac{1}{\sqrt[4]{\pi}\sqrt{\frac{2\sqrt{2}\pi k}{\omega n}}} e^{-\frac{(\frac{t-k\frac{2\pi}{\omega}/2}{2\sqrt{2}\pi k})^{2}}{2}} e^{-i\frac{2\sqrt{2}\pi k}{n}} \frac{t-k\frac{2\pi}{\omega}/2}{\frac{2\sqrt{2}\pi k}{\omega n}} \right) dt,$$
(29)

4. Obtain the damping ratio δ either by an *exact* or by a *closed-form* solution:

Exact: find the numerical root of (19)

$$M(n_1, n_2, k, \omega)_{Num} = e^{\frac{4\pi^2 k^2 \delta^2 \left(n_2^2 - n_1^2\right)}{n_1^2 n_2^2}} \sqrt{\frac{n_2}{n_1}} \frac{\operatorname{Erf}\left(\frac{2\pi k\delta}{n_1} + \frac{n_1}{4}\right) - \operatorname{Erf}\left(\frac{2\pi k\delta}{n_1} - \frac{n_1}{4}\right)}{\operatorname{Erf}\left(\frac{2\pi k\delta}{n_2} + \frac{n_2}{4}\right) - \operatorname{Erf}\left(\frac{2\pi k\delta}{n_2} - \frac{n_2}{4}\right)}$$
(30)

Closed-form: use Eq. (20) for a closed-form solution of the damping:

$$\delta_{Morlet} = \frac{n_1 n_2}{2 \pi \sqrt{k^2 n_2^2 - k^2 n_1^2}} \sqrt{\ln\left(\sqrt{\frac{n_1}{n_2}} M(n_1, n_2, k, \omega)_{Num}\right)}.$$
(31)



Figure 9 shows a noise-free damped signal that is 30 oscillations in length (k = 30).

Figure 9: Typical measured (noise-free) signal and the real part of the Morlet wavelet at $n_1 = 5$ and $n_2 = 10$.

Compared to damping identification with the CWT based on logarithmic decay, the damping identification using Eq.(20) is relatively easy to use and is not affected by the edge effect and requires significantly less computational time. In [9, 10], where CWT damping identification was researched, the computationally demanding CWT was calculated with a relatively dense time-frequency grid; even with the cross-sections damping identification method [4] that requires only one frequency point the CWT needs to be calculated with a relatively dense time grid. In [9, 10] the timegrid had more than 100 points. In this research the discussed Morlet-Wave Method requires similar computational operations as the CWT; however, twice at only one time-frequency point. Neglecting other numerical operations (the least-squares in the CWT and the root-finding in the Morlet-Wave method) the new method is approximately 50 or more times faster than the established CWT damping-identification methods. An exact analysis of the numerical load depends on the damping-identification parameters of the CWT and the wave method and is beyond the scope of this research. At the same time, the advantages of the damping-identification techniques with the CWT are preserved.

7. Conclusion

This study introduces a new viscous damping-identification method based on the Morlet wavelet. The method is a combination of the continuous wavelet transform and the finite integral of the Morlet wavelet. With the approximations given in the appendix, the Exact Morlet-wave (MW) dampingidentification method is obtained, and by further simplifying a Closed-Form, Morlet-wave, damping-identification method is possible. While preserving the CWT characteristics of the damping-identification of multi-degree-offreedom systems, the accuracy of the identification and the resistance to noise, the presented wave method is numerically significantly less demanding (approximately 50 times and more) and also completely overcomes the edge-effect problem of the CWT. For signals with a relatively high noise level, the root-finding Exact-MW identification method is needed; however, for signals with moderate noise the Closed-Form MW method gives an accurate solution.

Compared to the CWT based damping-identification the MW based dampingidentification hides the details of ridge/skeleton extraction and the regression analysis and therefore the MW based procedure is easily used as a "blackbox" method; this can be considered an advantage, but also a disadvantage as the misuse of the method is harder to identify.

In this research it was found that the Exact MW method damping-identification is accurate to within 10% at medium damped (damping ratio: 10^{-1} - 10^{-2}) signals with up to 5 dB noise; if the sampling rate is very high (50 to 100 times faster than the frequency analyzed) a 10% accuracy is reached, even for a very high level of noise (up to -5 dB). For signals with small damping (damping ratio: 10^{-3} - 10^{-4}) the 10% accuracy is reached at signals with less than 20 dB noise. Furthermore, for ultra-small damping (damping ratio: 10^{-5} - 10^{-6}) a 10% accuracy is only possible on a noise-free signal (120 dB). An investigation on closely spaced frequencies showed that the analysis is accurate within 20% at a single frequency-spread of the MW dampingidentification method, while at three frequency-spreads the accuracy is within 5%.

A comparison of the two presented methods, the Exact and the Closed-Form, showed that the Closed-Form method, while being simpler and giving closed-form solutions, is less resistant to noise (approximately 10 dB).

Appendix A. Finite Integral of Morlet Wave

In this section the complex integral given in Eq. (16):

$$I(n,k,\omega) = \int_0^T \left(X e^{-\delta \,\omega \,t} \cos(\omega \,t - \varphi) \right) \\ \left(\frac{1}{\sqrt[4]{\pi} \sqrt{\frac{2\sqrt{2}\pi k}{\omega \,n}}} e^{-\frac{\left(\frac{t-k}{2\omega}\frac{2\pi}{\omega}/2\right)^2}{2}} e^{-i\frac{2\sqrt{2}\pi k}{n}\frac{t-k}{\frac{2\pi}{\omega}/2}} \right) \, \text{(A.1)}$$

is discussed with the focus on finding the absolute value $|I(n, k, \omega)|$.

Using symbolic integration techniques, Eq. (A.1) can be rewritten as:

$$I(n,k,\omega) = \underbrace{-\left(\frac{\pi}{2}\right)^{3/4} X \sqrt{\frac{k}{n\,\omega}} e^{\frac{\pi k \left(4\,\pi k\,\delta^2 - n^2\left(i+\delta\right)\right)}{n^2}}_{A} (B\,C+D),\qquad(A.2)$$

where:

$$B = e^{\frac{16\pi^2 k^2 (1+i\delta)}{n^2} - i\varphi}$$
(A.3)

$$C = \operatorname{Erf}\left(\frac{2\pi k\,\delta}{n} - \frac{n}{4}\right) - \operatorname{Erf}\left(\frac{2\pi k\,\delta}{n} + \frac{n}{4}\right) \tag{A.4}$$

$$D = e^{i\varphi} \left(\operatorname{Erf}\left(-\frac{n^2 - 8\pi k (\delta - 2i)}{4n}\right) - \operatorname{Erf}\left(\frac{n^2 + 8\pi k (\delta - 2i)}{4n}\right) A.5\right)$$

For typical values of the time-spread parameter n, the number of oscillations k and the damping ratio δ the Eq.(A.2) results in numerically manageable numbers; however, the part B results in very high numbers that are numerically impossible to deal with. Consequentially, the main goal of this section is to simplify and approximate the absolute value $|I(n, k, \omega)|$ for typical parameters: The absolute value $|I(n, k, \omega)|$ is (A.1):

$$|I(n,k,\omega)| = |A(BC+D)| = |A||BC+D|$$
(A.6)

 $|I(n,k,\omega)|$ can be approximated with:

$$|I(n,k,\omega)| \approx |A| |B| |C|, \tag{A.7}$$

if the following assumption is true:

$$|D| \ll |B| |C|. \tag{A.8}$$

To prove Eq.(A.8) the Error function Erf is expanded into a Taylor powerseries. By assuming that for the Error function $\operatorname{Erf}(x)$ the value $x \gg 0$, then the power-series expansions around ∞ or $-\infty$ are:

$$\operatorname{Erf}(x)|_{x \to \pm \infty} = \pm 1 + e^{-x^2} \left(-\frac{1}{\sqrt{\pi} x} + \frac{1}{2\sqrt{\pi} x^3} + \cdots \right)$$
 (A.9)

By using the first-order expansion $\operatorname{Erf}(x)|_{x\to\pm\infty} = \pm 1 + e^{-x^2} \left(-\frac{1}{\sqrt{\pi}x}\right)$ the Eqs.(A.4,A.5) can be rewritten as:

$$C = -2 + \frac{4n}{\sqrt{\pi}} \left(\frac{e^{-\left(\frac{n}{4} - \frac{2\pi k \,\delta}{n}\right)^2}}{n^2 - 8\pi k \,\delta} + \frac{e^{-\left(\frac{n}{4} + \frac{2\pi k \,\delta}{n}\right)^2}}{n^2 + 8\pi k \,\delta} \right)$$
(A.10)
$$D = e^{\mathbf{i}\,\varphi} \left(-2 + \frac{4n}{\sqrt{\pi}} \left(\frac{e^{-\left(\frac{n^2 - 8\pi k \,(\delta - 2\,\mathbf{i})}{4n}\right)^2}}{n^2 - 8\pi k \,(\delta - 2\,\mathbf{i})} + \frac{e^{-\left(\frac{n^2 + 8\pi k \,(\delta - 2\,\mathbf{i})}{4n}\right)^2}}{n^2 + 8\pi k \,(\delta - 2\,\mathbf{i})} \right) \right)$$
(A.11)

By assuming real positive values it is relatively easy to deduce:

$$|B| = e^{\frac{16\pi^2 k^2}{n^2}} \tag{A.12}$$

and by further assuming:

$$\frac{n}{4} - \frac{2\pi k\,\delta}{n} > 0 \tag{A.13}$$

the absolute value of ${\cal C}$ reduces to:

$$|C| = 2 - \frac{4 n e^{-\frac{\left(8 \pi k \,\delta + n^2\right)^2}{16 \,n^2}} \left(n^2 \left(e^{2 \pi k \,\delta} + 1\right) + 8 \pi k \,\delta \left(e^{2 \pi k \,\delta} - 1\right)\right)}{\sqrt{\pi} \left(n^4 - 64 \,\pi^2 \,k^2 \,\delta^2\right)}$$
(A.14)

Continuing with the absolute value of D:

$$|D| = \frac{4n}{\sqrt{\pi}} \left| \underbrace{\frac{e^{-\left(\frac{n^2 - 8\pi k (\delta - 2i)}{4n}\right)^2}}{\frac{n^2 - 8\pi k (\delta - 2i)}{D_1}}}_{D_1} + \underbrace{\frac{e^{-\left(\frac{n^2 + 8\pi k (\delta - 2i)}{4n}\right)^2}}{\frac{n^2 + 8\pi k (\delta - 2i)}{D_2}}}_{D_2} \right| - 2$$
(A.15)

By assuming that $|D| \gg 2$ the value 2 can be neglected. Furthermore, by using $|D_1 + D_2| \le |D_1| + |D_2|$, the value |D| can be limited by:

$$|D| \le \frac{4n}{\sqrt{\pi}} (|D_1| + |D_2|) \tag{A.16}$$



Figure A.10: $|B| |C| / |D^*|$ vs. δ for typical parameter.

assuming real positive values, Eq.(A.16) can be deduced to:

$$|D| \le \frac{4n \left(\frac{1}{\sqrt{256 \pi^2 k^2 + (8\pi k \,\delta + n^2)^2}} + \frac{e^{2\pi k \,\delta}}{\sqrt{256 \pi^2 k^2 + (n^2 - 8\pi k \,\delta)^2}}\right) e^{-\frac{\left(8\pi k \,(\delta - 2) + n^2\right)\left(8\pi k \,(\delta + 2) + n^2\right)}{16 \,n^2}}}{\sqrt{\pi}}$$
(A.17)

Fig. A.10 shows the $|B| |C| / |D^*|$ versus the damping factor for typical values: $n = 10, k = 30 (|D^*|$ is the upper limit of |D| Eq.(A.17)); it is clear that the assumption of Eq.(A.8) is valid only if the assumption (A.13) is valid. From Eq. (A.13) it follows that for a selected time-spread parameter n and the number of oscillations k the theoretically identified damping ratio is limited by (A.13):

$$\delta \le \frac{n^2}{8 \pi k}.\tag{A.18}$$

Finally, it follows from (A.7)

$$|I(n,k,\omega)| \approx X \left(\frac{\pi}{4}\right)^{4/3} \sqrt{\frac{k}{n\,\omega}} e^{\frac{\pi\,k\,\delta\left(4\,\pi\,k\,\delta-n^2\right)}{n^2}} \left(\operatorname{Erf}\left(\frac{2\,\pi\,k\,\delta}{n} + \frac{n}{4}\right) - \operatorname{Erf}\left(\frac{2\,\pi\,k\,\delta}{n} - \frac{n}{4}\right)\right),$$
(A.19)

where |C| was deduced from Eq. (A.4) and |A| in Eq. (A.7) assuming real positive values was simplified to:

$$|A| = X \left(\frac{\pi}{2}\right)^{3/4} e^{\frac{\pi k \left(4 \pi k \left(\delta^2 - 4\right) - n^2 \delta\right)}{n^2}} \sqrt{\frac{k}{n \,\omega}} \tag{A.20}$$

References

- W. Staszewski, Identification of damping in mdof systems using timescale decomposition, Journal of Sound and Vibration 203 (2) (1997) 283–305.
- [2] M. Prandina, J. E. Mottershead, E. Bonisoli, An assessment of damping identification methods, Journal of Sound and Vibration 323 (3-5) (2009) 662–676. doi:10.1016/j.jsv.2009.01.022.
- [3] F. Magalhaes, A. Cunha, E. Caetano, R. Brincker, Damping estimation using free decays and ambient vibration tests, Mechanical Systems and Signal Processing In Press, Corrected Proof (2009) –. doi:DOI: 10.1016/j.ymssp.2009.02.011.
- [4] W. Staszewski, Identification of non-linear systems using multi-scale ridges and sceletons of the wavelet transform, Journal of Sound and Vibration 214 (4) (1998) 639–658.
- [5] M. Ruzzene, A. Fasana, L. Garibaldi, B. Piombo, Natural frequencies and dampings identification using wavelet transform: Application to real data, Mechanical Systems and Signal Processing 11 (2) (1997) 207– 218.
- [6] C. H. Lamarque, S. Pernot, A. Cuer, Damping identification in multidegree-of-freedom systems via a wavelet-logarithmic decrement-part 1:theory, Journal of Sound and Vibration 235 (3) (2000) 361–374.
- [7] S. Hans, E. Ibraim, S. Pernot, C. Boutin, C. H. Lamarque, Damping identification in multi-degree-of-freedom systems via a waveletlogarithmic decrement-part 2:study of a civil engineering building, Journal of Sound and Vibration 235 (3) (2000) 375–403.
- [8] R. Ghanem, F. Romeo, A wavelet-based approach for the identification of linear time-varying dynamical systems, Journal of Sound and Vibration 234 (4) (2000) 555–576.
- [9] J. Slavič, I. Simonovski, M. Boltežar, Damping identification using a continuous wavelet transform: Application to real data, Journal of Sound and Vibration 262 (2003) 291–307.
- [10] M. Boltežar, J. Slavič, Enhancements to the continuous wavelet transform for damping identifications on short signals, Mechanical Systems and Signal Processing 18 (5) (2004) 1065–1076.

- [11] J. Lardies, S. Gouttebroze, Identification of modal parameters using the wavelet transform, International Journal of Mechanical Sciences 44 (11) (2002) 2263–2283. doi:10.1016/S0020-7403(02)00175-3.
- [12] J. Lardies, M. Ta, M. Berthillier, Modal parameter estimation based on the wavelet transform of output data, Archive of Applied Mechanics 73 (9-10) (2004) 718–733. doi:10.1007/s00419-004-0329-6.
- [13] T. Le, P. Argoul, Continuous wavelet transform for modal identification using free decay response, Journal of Sound and Vibration 277 (1-2) (2004) 73–100. doi:10.1016/j.jsv.2003.08.049.
- [14] H. Yin, D. Duhamel, P. Argoul, Natural frequencies and damping estimation using wavelet transform of a frequency response function, Journal of Sound and Vibration 271 (3-5) (2004) 999–1014. doi:10.1016/j.jsv.2003.03.002.
- [15] S. Erlicher, P. Argoul, Modal identification of linear non-proportionally damped systems by wavelet transform, Mechanical Systems and Signal Processing 21 (3) (2007) 1386–1421. doi:10.1016/j.ymssp.2006.03.010.
- [16] M. Česnik, J. Slavič, M. Boltežar, Spatial-Mode-Shape Identification using a Continuous Wavelet Transform, Strojniski vestnik-Journal of Mechanical Engineering 55 (5) (2009) 277–285.
- [17] S.-L. Chen, J.-J. Liu, H.-C. Lai, Wavelet analysis for identification of damping ratios and natural frequencies, Journal of Sound and Vibration 323 (1-2) (2009) 130 – 147. doi:DOI: 10.1016/j.jsv.2009.01.029.
- [18] B. Yan, A. Miyamoto, A comparative study of modal parameter identification based on wavelet and Hilbert-Huang transforms, Computer-Aided Civil and Infrastructure Engineering 21 (1) (2006) 9–23.
- [19] Z. Li, M. Crocker, A study of joint time-frequency analysis-based modal analysis, IEEE Transactions on Instrumentation and Measurement 55 (6) (2006) 2335 – 2342. doi:DOI: 10.1109/TIM.2006.884137.
- [20] J.-B. Tan, Y. Liu, L. Wang, W.-G. Yang, Identification of modal parameters of a system with high damping and closely spaced modes by combining continuous wavelet transform with pattern search, Mechanical Systems and Signal Processing 22 (5) (2008) 1055–1060. doi:10.1016/j.ymssp.2007.11.017.

- [21] Z. Yan, A. Miyamoto, Z. Jiang, Frequency slice wavelet transform for transient vibration response analysis, Mechanical Systems and Signal Processing 23 (5) (2009) 1474–1489. doi:10.1016/j.ymssp.2009.01.008.
- [22] Z. Yan, A. Miyamoto, Z. Jiang, X. Liu, An overall theoretical description of frequency slice wavelet transform, Mechanical Systems and Signal Processing 24 (2) (2010) 491–507. doi:10.1016/j.ymssp.2009.07.002.
- [23] A. Grossmann, J. Morlet, Decomposition of hardy function into square integrable wavelets of constant shape, SIAM Journal of Mathematical Analysis and Applications 15 (4) (1984) 723–736.
- [24] S. Mallat, A Wavelet Tour of Signal Processing, 2nd Edition, Academic Press, 1999.
- [25] I. Simonovski, M. Boltežar, The norms and variances of the gabor, morlet and general harmonic wavelet functions, Journal of Sound and Vibration 264 (3) (2003) 545–557.
- [26] W. Thomson, Theory of Vibration with Applications, 4th Edition, London: Chapman and Hall, 1993.
- [27] C. Beards, Structural Vibration: Analysis and Damping, Arnold, 1996.
- [28] J. Tukey, Exploratory Data Analysis, Addison-Wesley, Reading, MA, 1977.